5-3 Solving Quadratic Equations by Factoring

Main Ideas

- Write quadratic equations in intercept form.
- Solve quadratic equations by factoring.

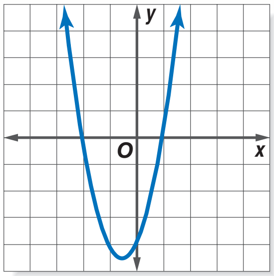
New Vocabulary

intercept form

FOIL method

GET READY for the Lesson

The **intercept form** of a quadratic equation is $y = a(x - p)(x - q)$. In the equation, p and q represent the x-intercepts of the graph corresponding to the equation. The intercept form of the equation shown in the graph is $y = 2(x - 1)(x + 2)$. The x-intercepts of the graph are 1 and -2. The standard form of the equation is $y = 2x^2 + 2x - 4$.



Intercept Form Changing a quadratic equation in intercept form to standard form requires the use of the FOIL method. The **FOIL method** uses the Distributive Property to multiply binomials.

KEY CONCEPT — FOIL Method for Multiplying Binomials

The product of two binomials is the sum of the products of **F** the *first* terms, **O** the *outer* terms, **I** the *inner* terms, and **L** the *last* terms.

To change $y = 2(x - 1)(x + 2)$ to standard form, use the FOIL method to find the product of $(x - 1)$ and $(x + 2)$, $x^2 + x - 2$, and then multiply by 2. The standard form of the equation is $y = 2x^2 + 2x - 4$.

You have seen that a quadratic equation of the form $(x - p)(x - q) = 0$ has roots p and q. You can use this pattern to find a quadratic equation for a given pair of roots.

EXAMPLE 1 Write an Equation Given Roots

Write a quadratic equation with $\frac{1}{2}$ and -5 as its roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers.

$(x - p)(x - q) = 0$	Write the pattern.
$\left(x - \frac{1}{2}\right)[x - (-5)] = 0$	Replace p with $\frac{1}{2}$ and q with -5.
$\left(x - \frac{1}{2}\right)(x + 5) = 0$	Simplify.
$x^2 + \frac{9}{2}x - \frac{5}{2} = 0$	Use FOIL.
$2x^2 + 9x - 5 = 0$	Multiply each side by 2 so that b and c are integers.

Study Tip

Writing an Equation

The pattern $(x - p)(x - q) = 0$ produces one equation with roots p and q.

In fact, there are an infinite number of equations that have these same roots.

CHECK Your Progress

1. Write a quadratic equation with $-\frac{1}{3}$ and 4 as its roots. Write the equation in standard form.

Solve Equations by Factoring In the last lesson, you learned to solve a quadratic equation by graphing. Another way to solve a quadratic equation is by factoring an equation in standard form. When an equation in standard form is factored and written in intercept form $y = a(x - p)(x - q)$, the solutions of the equation are p and q.

The following factoring techniques, or patterns, will help you factor polynomials. Then you can use the Zero Product Property to solve equations.

CONCEPT SUMMARY *Factoring Techniques*

Factoring Technique	General Case
Greatest Common Factor (GCF)	$a^3b^2 - 3ab^2 = ab^2(a^2 - 3)$
Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$
Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
General Trinomials	$acx^2 + (ad + bc)x + bd =$ $(ax + b)(cx + d)$

The FOIL method can help you factor a polynomial into the product of two binomials. Study the following example.

$$(ax + b)(cx + d) = \overbrace{ax \cdot cx}^{F} + \overbrace{ax \cdot d}^{O} + \overbrace{b \cdot cx}^{I} + \overbrace{b \cdot d}^{L}$$
$$= acx^2 + (ad + bc)x + bd$$

Notice that the product of the coefficient of x^2 and the constant term is $abcd$. The product of the two terms in the coefficient of x is also $abcd$.

EXAMPLE 2 Two or Three Terms

Factor each polynomial.

a. $5x^2 - 13x + 6$

To find the coefficients of the x-terms, you must find two numbers with a product of $5 \cdot 6$ or 30, and a sum of -13. The two coefficients must be -10 and -3 since $(-10)(-3) = 30$ and $-10 + (-3) = -13$.

Rewrite the expression using $-10x$ and $-3x$ in place of $-13x$ and factor by grouping.

$5x^2 - 13x + 6 = 5x^2 - 10x - 3x + 6$ Substitute $-10x - 3x$ for $-13x$.

$= (5x^2 - 10x) + (-3x + 6)$ Associative Property

$= 5x(x - 2) - 3(x - 2)$ Factor out the GCF of each group.

$= (5x - 3)(x - 2)$ Distributive Property

Study Tip

The difference of two squares should always be done before the difference of two cubes. This will make the next step of the factorization easier.

b. $m^6 - n^6$

$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$ Difference of two squares

$= (m + n)(m^2 - mn + n^2)(m - n)(m^2 + mn + n^2)$ Sum and difference of two cubes

CHECK Your Progress

2A. $3xy^2 - 48x$ **2B.** $c^3d^3 + 27$

Solving quadratic equations by factoring is an application of the **Zero Product Property**.

KEY CONCEPT *Zero Product Property*

Words For any real numbers a and b, if $ab = 0$, then either $a = 0$, $b = 0$, or both a and b equal zero.

Example If $(x + 5)(x - 7) = 0$, then $x + 5 = 0$ or $x - 7 = 0$.

EXAMPLE Two Roots

3 **Solve $x^2 = 6x$ by factoring. Then graph.**

$x^2 = 6x$	Original equation
$x^2 - 6x = 0$	Subtract $6x$ from each side.
$x(x - 6) = 0$	Factor the binomial.
$x = 0$ or $x - 6 = 0$	Zero Product Property
$x = 6$	Solve the second equation.

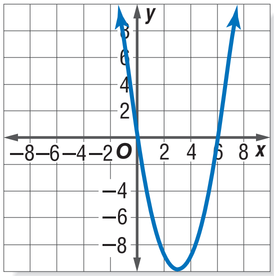
The solution set is $\{0, 6\}$.

To complete the graph, find the vertex. Use the equation for the axis of symmetry.

$x = -\frac{b}{2a}$	Equation of the axis of symmmetry
$= -\frac{-6}{2}(1)$	$a = 1, b = -6$
$= 3$	Simplify.

Therefore, the x-coordinate of the vertex is 3. Substitute 3 into the equation to find the y-value.

$y = x^2 - 6x$	Original equation
$= 3^2 - 6(3)$	$x = 3$
$= 9 - 18$	Simplify.
$= -9$	Subtract.



The vertex is at $(3, -9)$. Graph the x-intercepts $(0, 0)$ and $(6, 0)$ and the vertex $(3, -9)$, connecting them with a smooth curve.

CHECK Your Progress **Solve each equation by factoring. Then graph.**

3A. $3x^2 = 9x$ **3B.** $6x^2 = 1 - x$

Personal Tutor at algebra2.com

Study Tip

Double Roots

The application of the Zero Product Property produced two identical equations, $x - 8 = 0$, both of which have a root of 8. For this reason, 8 is called the *double root* of the equation.

EXAMPLE Double Root

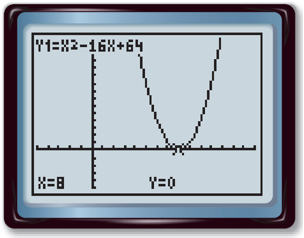
4 **Solve $x^2 - 16x + 64 = 0$ by factoring.**

$x^2 - 16x + 64 = 0$	Original equation
$(x - 8)(x - 8) = 0$	Factor.
$x - 8 = 0$ or $x - 8 = 0$	Zero Product Property
$x = 8$ $\quad$ $x = 8$	Solve each equation.

The solution set is $\{8\}$.

(continued on the next page)

CHECK The graph of the related function, $f(x) = x^2 - 16x + 64$, intersects the x-axis only once. Since the zero of the function is 8, the solution of the related equation is 8.



CHECK Your Progress

Solve each equation by factoring.

4A. $x^2 + 12x + 36 = 0$ **4B.** $x^2 - 25 = 0$

CHECK Your Understanding

Example 1 (p. 253)

Write a quadratic equation with the given root(s). Write the equation in standard form.

1. $-4, 7$ **2.** $\frac{1}{2}, \frac{4}{3}$ **3.** $-\frac{3}{5}, -\frac{1}{3}$

Example 2 (p. 254)

Factor each polynomial.

4. $x^3 - 27$ **5.** $4xy^2 - 16x$ **6.** $3x^2 + 8x + 5$

Examples 3, 4 (pp. 255–256)

Solve each equation by factoring. Then graph.

7. $x^2 - 11x = 0$ **8.** $x^2 + 6x - 16 = 0$ **9.** $4x^2 - 13x = 12$

10. $x^2 - 14x = -49$ **11.** $x^2 + 9 = 6x$ **12.** $x^2 - 3x = -\frac{9}{4}$

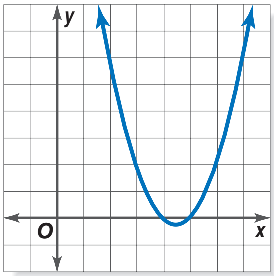
Exercises

HOMEWORK HELP

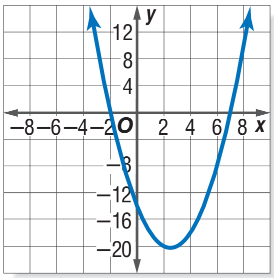
For Exercises	See Examples
13–16	1
17–20	2
21–32	3, 4

Write a quadratic equation in standard form for each graph.

13.



14.



Write a quadratic equation in standard form with the given roots.

15. $4, -5$ **16.** $-6, -8$

Factor each polynomial.

17. $x^2 - 7x + 6$ **18.** $x^2 + 8x - 9$

19. $3x^2 + 12x - 63$ **20.** $5x^2 - 80$

Solve each equation by factoring. Then graph.

21. $x^2 + 5x - 24 = 0$ **22.** $x^2 - 3x - 28 = 0$

23. $x^2 = 25$ **24.** $x^2 = 81$

25. $x^2 + 3x = 18$ **26.** $x^2 - 4x = 21$

27. $-2x^2 + 12x - 16 = 0$ **28.** $-3x^2 - 6x + 9 = 0$

29. $x^2 + 36 = 12x$ **30.** $x^2 + 64 = 16x$

31. NUMBER THEORY Find two consecutive even integers with a product of 224.

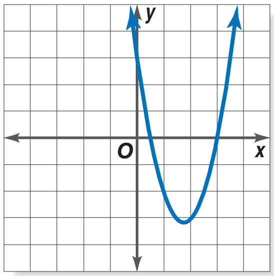
32. **PHOTOGRAPHY** A rectangular photograph is 8 centimeters wide and 12 centimeters long. The photograph is enlarged by increasing the length and width by an equal amount in order to double its area. What are the dimensions of the new photograph?

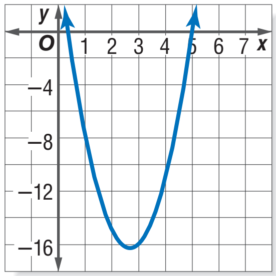
Solve each equation by factoring.

33. $3x^2 = 5x$
34. $4x^2 = -3x$
35. $4x^2 + 7x = 2$
36. $4x^2 - 17x = -4$
37. $4x^2 + 8x = -3$
38. $6x^2 + 6 = -13x$
39. $9x^2 + 30x = -16$
40. $16x^2 - 48x = -27$

41. Find the roots of $x(x + 6)(x - 5) = 0$.
42. Solve $x^3 = 9x$ by factoring.

Write a quadratic equation with the given graph or roots.

43. 

44. 

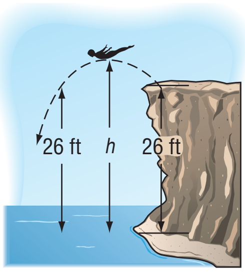
45. $-\frac{2}{3}, \frac{3}{4}$
46. $-\frac{3}{2}, -\frac{4}{5}$

Real-World Link

A board foot is a measure of lumber volume. One piece of lumber 1 foot long by 1 foot wide by 1 inch thick measures one board foot.

Source: www.wood-worker.com

47. **DIVING** To avoid hitting any rocks below, a cliff diver jumps up and out. The equation $h = -16t^2 + 4t + 26$ describes her height h in feet t seconds after jumping. Find the time at which she returns to a height of 26 feet.



FORESTRY For Exercises 48 and 49, use the following information.
Lumber companies need to be able to estimate the number of board feet that a given log will yield. One of the most commonly used formulas for estimating board feet is the *Doyle Log Rule*, $B = \frac{L}{16}(D^2 - 8D + 16)$ where B is the number of board feet, D is the diameter in inches, and L is the length of the log in feet.

48. Rewrite Doyle's formula for logs that are 16 feet long.
49. Find the root(s) of the quadratic equation you wrote in Exercise 48. What do the root(s) tell you about the kinds of logs for which Doyle's rule makes sense?

EXTRA PRACTICE
See pages 900, 930.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

50. **FIND THE ERROR** Lina and Kristin are solving $x^2 + 2x = 8$. Who is correct? Explain your reasoning.

Lina	Kristin
$x^2 + 2x = 8$	$x^2 + 2x = 8$
$x(x + 2) = 8$	$x^2 + 2x - 8 = 0$
$x = 8$ or $x + 2 = 8$	$(x + 4)(x - 2) = 0$
$x = 6$	$x + 4 = 0$ or $x - 2 = 0$
	$x = -4$ $\quad$ $x = 2$

51. **OPEN ENDED** Choose two integers. Then write an equation with those roots in standard form. How would the equation change if the signs of the two roots were switched?

52. **CHALLENGE** For a quadratic equation of the form $(x - p)(x - q) = 0$, show that the axis of symmetry of the related quadratic function is located halfway between the x-intercepts p and q.

53. ***Writing in Math*** Use the information on page 253 to explain how to solve a quadratic equation using the Zero Product Property. Explain why you cannot solve $x(x + 5) = 24$ by solving $x = 24$ and $x + 5 = 24$.

STANDARDIZED TEST PRACTICE

54. **ACT/SAT** Which quadratic equation has roots $\frac{1}{2}$ and $\frac{1}{3}$?

A $5x^2 - 5x - 2 = 0$

B $5x^2 - 5x + 1 = 0$

C $6x^2 + 5x - 1 = 0$

D $6x^2 - 5x + 1 = 0$

55. **REVIEW** What is the solution set for the equation $3(4x + 1)^2 = 48$?

F $\left\{\frac{5}{4}, -\frac{3}{4}\right\}$

G $\left\{-\frac{5}{4}, \frac{3}{4}\right\}$

H $\left\{\frac{15}{4}, -\frac{17}{4}\right\}$

J $\left\{\frac{1}{3}, -\frac{4}{3}\right\}$

Spiral Review

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located. (Lesson 5-2)

56. $0 = -x^2 - 4x + 5$

57. $0 = 4x^2 + 4x + 1$

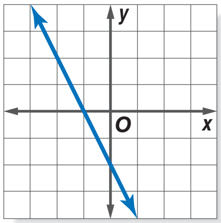
58. $0 = 3x^2 - 10x - 4$

59. Determine whether $f(x) = 3x^2 - 12x - 7$ has a maximum or a minimum value. Then find the maximum or minimum value. (Lesson 5-1)

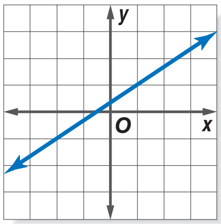
60. **CAR MAINTENANCE** Vince needs 12 quarts of a 60% anti-freeze solution. He will combine an amount of 100% anti-freeze with an amount of a 50% anti-freeze solution. How many quarts of each solution should be mixed to make the required amount of the 60% anti-freeze solution? (Lesson 4-8)

Write an equation in slope-intercept form for each graph. (Lesson 2-4)

61.



62.



GET READY for the Next Lesson

PREREQUISITE SKILL Name the property illustrated by each equation. (Lesson 1-2)

63. $2x + 4y + 3z = 2x + 3z + 4y$

64. $3(6x - 7y) = 3(6x) + 3(-7y)$

65. $(3 + 4) + x = 3 + (4 + x)$

66. $(5x)(-3y)(6) = (-3y)(6)(5x)$